HISTORICAL PV OF DEBT IN DEVELOPING COUNTRIES, 1980-2002

Long-term debt sustainability analyses are usually based on nominal debt numbers. While nominal debt is an important debt indicator, the present value of debt payments [PV] is in many respects considered superior. However, there have been no PV estimates in the literature spanning long enough time periods and using consistently concurrent discount rates. This paper is intended to fill this gap: it develops a methodology for historical PV computations and applies it to the actual debt data.

The computation of PV is based on inputs from the Debt Reporting System (DRS) maintained by DECDG Department of the World Bank. The DRS is a unique depository of historical debt data for developing countries that allows data mining, projections and detailed analysis. However, it is extremely time-consuming to analyze its in-depth loan-by-loan information. This paper pursues a different approach: instead of loan-by-loan data, in PV estimates a somewhat more aggregated dataset is used. It turns out to be possible to utilize average terms for loan aggregates for the loans with similar terms (average grace, maturity and interest rate) to achieve sufficiently precise PV estimates for the aggregate.  

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1. DRS Data Sources and Aggregation Methodology

The World Bank is the sole repository for statistics on the external debt of developing countries on a loan-by-loan basis. The Debtor Reporting System (DRS) was set up in 1951 to monitor these statistics. All DRS data that are used in the PV calculations are reported directly by the debtors except for lending by some multilateral agencies in which case data are taken from the creditors’ records. These creditors include the African Development Bank, the Asian Development Bank, the International Monetary Fund (IMF), the Inter-American Development Bank, the International Bank for Reconstruction and Development (IBRD), and the International Development Bank (IDA). (The IBRD and IDA are components of the World Bank).

The DRS data is highly detailed and comprehensive. The loan-by-loan data includes private, public, foreign, short term, long term, privately-guaranteed, and publicly-guaranteed debt. In order to facilitate data manipulation and intermediate computations, a smaller subset of the DRS data was extracted for PV calculations. The DRS loan-by-loan data was aggregated by debtor country, pay currency, effective date [commitment date] and creditor type. In addition, the scope of analysis was limited to long term public and publicly-guaranteed [PPG] foreign debt. This specific dataset is then processed using terms for average grace period, maturity, and interest rate. The PV calculations are performed by applying average terms to the loan aggregates with similar terms utilizing the process detailed in this paper.

2. Derivation of PV

We start with definitions: Present Value (PV) is the discounted value of a payment or stream of payments to be received in the future, taking into consideration a specific interest or discount rate. Present Value represents a series of future cash flows expressed in today's currency units.

Or, put differently, with obvious notation:

\[ PV = \sum_{t=t_0}^{t_n} \frac{DS_t}{(1 + d)^{t-t_0}} \]  

(1.1)

The continuous version of the formula will be, respectively:
\[(1.2) \quad PV = \int_{t_0}^{t_5} ds(t)e^{d(t_5-t)} dt\]

again, with obvious notation, where DS stands for debt service, and d stands for discount rate. The formula (1.1) can be extended to cover monthly and daily [and any arbitrary period] discounting.

From this presentation, it follows that the PV has the following properties:

(1) linearity wrt. debt service [multiplying debt service by a factor will result in the PV greater than the PV of the original debt service by the same factor];
(2) monotonicity wrt. debt service [greater debt service results in greater PV];
(3) additivity wrt. debt service if discount rates do not change [i.e. the sum of PVs of individual debt flows is equal to the PV of the sum of individual debt flows; or, put differently, in order to compute the PV of a debt portfolio, it is enough to group debt service into currency groups /or other groups with the same discount rate/ and compute PV for those groups only]

The PV is equal to the nominal [face] value of a loan if and only if the discount rate is equal to the interest rate. In all other cases there will be a discrepancy between the nominal (face) value of a loan and its PV. The difference is usually called the Grant Element of the loan.

PV is often considered the single most important indicator of indebtedness used in debt analysis. Commercial banks use it in most of their operations. It is preferred over debt outstanding and simple sum of debt repayments because these categories do not account for differences in repayment terms.

Accordingly, most of the debt operations, including those of Paris Club, are tied to the PV (or, as it is sometimes [incorrectly] called, the NPV).

3. Properties of the Present Value

The most important property of the PV concept is equality of PV and face value of debt (initial DOD) whenever interest rate is constant and equal to discount rate. We will provide a proof of this property both for discreet and continuous cases below:

\textit{Notation}: 

**Discrete Case:**

dod stands for debt outstanding and disbursed,
inp – interest payments
prp – principal payments
tds – total debt service

**Continuous Case:**
p – principal payment flow
y – debt outstanding and disbursed
ν - interest payment flow
i – interest rate

the first time period is 0, the last one is N.

**Discrete Case:**

\[
dod_t = dod_{t-1} - prp_t \\
inp_t = i \cdot dod_{t-1} \\
tds_t = i \cdot dod_{t-1} + prp_t \\
tds_t = i \cdot dod_{t-1} - dod_t + dod_{t-1} \\
tds_t = (i+1) \cdot dod_{t-1} - dod_t \\
\]

\[
dod_t = dod_0 - \sum_{1}^{i} prp_t \\
\sum_{1}^{N} tds_t = \sum_{1}^{N} [(i+1) \cdot dod_{t-1} - dod_t] \\
PV = \sum_{1}^{N} \frac{tds_t}{(1+i)^t} = \sum_{1}^{N} \frac{(i+1) \cdot dod_{t-1} - dod_t}{(1+i)^t} = \\
= \sum_{1}^{N} \left[ \frac{dod_{t-1}}{(1+i)^{t-1}} - \frac{dod_t}{(1+i)^t} \right] = dod_0 - \frac{dod_N}{(1+i)^N} \\
PV \equiv dod_0, \text{ as } dod_N \text{ is equal to zero}
\]

**Continuous Case:**
\(-p = \dot{y}\) (principal payments)
\[\nu = i \cdot y\] (interest payments)

hence

\[
PV = \int_{t=t_0}^{t=t_N} (-\dot{y} + i \cdot y) e^{-it} \, dt, \text{ or}
\]

\[
PV = \int_{t=t_0}^{t=t_N} d(-ye^{-it}) = -ye^{-it} \bigg|_{t_0}^{t_N} = y(t_0),
\]

as \(y(t_N)\) is equal to zero.

Related to the \(PV\) is the concept of \textit{Grant Element} (GE). The \textit{Grant Element} of a loan is the difference between its \(PV\) as computed using actual interest rates and using some accepted discount factor (such as CIRR or another market discount rate).

Grant Element reflects financial terms of commitments: interest rate, maturity and grace period (interval to first repayment of capital). It measures the concessionality of a loan, as the difference between present value at a particular interest rate versus that at the market rate. For example, the market rate is assumed to be 10 per cent in DAC statistics. Thus, the Grant Element is nil for a loan carrying an interest rate of 10 per cent; it is 100 per cent for a grant; and it lies anywhere in between these two limits for a soft loan. If the face value of a loan is multiplied by its Grant Element, the result is referred to as the grant equivalent of that loan.

\[
PV = \sum_{t=1}^{N} \frac{tds_t}{(1+d)^t} = \sum_{t=1}^{N} \frac{(d-\Delta+1) \cdot dod_{t-1} - dod_t}{(1+d)^t} =
\]

\[
= \sum_{t=1}^{N} (1+d) \cdot dod_{t-1} - dod_t \left(\frac{1}{(1+d)^t}\right) - \sum_{t=1}^{N} \frac{\Delta dod_{t-1}}{(1+d)^t} =
\]

\[
= dod_0 - \Delta \sum_{t=1}^{N} \frac{dod_{t-1}}{(1+d)^t}, \text{ where } \Delta \text{ is the difference}
\]

between interest and discount rates. And Grant Element is:

\[
GE = \Delta \sum_{t=1}^{N} \frac{dod_{t-1}}{(1+d)^t}.
\]

Interestingly, the GE is thus exactly proportional to the difference between interest and discount rates. Operationally, it means that one can compute effect of interest rate changes on \(PV\) quite simply. Also, once \(GE\) for any given interest rate is known, one can always reestimate \(GE\) for any other interest rate value using proportional change in interest rate (for any given discount rate):
Now, using a finite series formula we can express PV and GE of a loan:

\[
GE(i_1) / GE(i_2) = \frac{i_1 - d}{i_2 - d}
\]

\[
\sum_{0}^{n-1} (a + kr) q^i = a - \frac{a + (n-1) r}{1 - q} + \frac{rq(1 - q^{n-1})}{(1 - q)^2}
\]

\[
GE = \Delta \sum_{1}^{N} \frac{dod_{t-1} / dod_0}{(1 + d)^t}
\]

\[
dod_t = dod_0 - t \frac{dod_0}{m}, \text{ where } m \text{ is principal repayment length}
\]

and \( q = 1/(1 + d) \)

then,

\[
GE = \Delta q \sum_{0}^{N-1} \frac{1-t/m}{(1 + d)^t} = \Delta q \sum_{0}^{N-1} (1-t/m)q^t =
\]

\[
= \Delta q \sum_{0}^{N-m} q^t + \Delta q^{N-m+1} \sum_{0}^{m-1} (1-t/m)q^t =
\]

\[
= \Delta q\left[ \frac{1-q^{N-m}}{1-q} + q^{N-m} \sum_{0}^{m-1} (1-t/m)q^t \right], \text{ or}
\]

\[
GE = \frac{i-d}{d^2 m} \left[ dm - \left( \frac{1}{1+d} \right)^{N-m} + \left( \frac{1}{1+d} \right)^N \right]
\]

and, respectively, PV can be expressed as follows:

\[
PV = dod_0 (1 - GE)
\]

As we can see, the above formulas allow efficient calculation of the PV and GE of a loan without going through tedious year-by-year computations.

All of the above terms can be obtained from the DRS database. Moreover, it is also possible to obtain the GE estimates based on the DAC’s 10% discount rate, which could be used to correct our estimated values. So the overall formula used in the estimations would be:

\[
GE_{final} = GE + \Delta
\]

where

\[
GE = \frac{i-d}{d^2 m} \left[ dm - \left( \frac{1}{1+d} \right)^{N-m} + \left( \frac{1}{1+d} \right)^N \right], \text{ and}
\]

\[
\Delta = GE_{estimated by DRS at 10%} - \frac{i-0.1}{0.1^2 m} \left[ dm - \left( \frac{1}{1+0.1} \right)^{N-m} + \left( \frac{1}{1+0.1} \right)^N \right]
\]
Limit cases of the GE function are interesting per se [see Annex for details]. Several cases that have operational significance are discussed, including ones where $i \to 0$ (interest rate approaching 0), $d \to 0$ (discount rate approaching 0), $n \to M$ (length of payments approaching maturity, i.e. grace approaching zero), $m \to 0$ (length of payments approaching 0, i.e. bullet payment). These cases allow for very simple presentations.

4. Some results of computations of PV using DRS data

The results are presented in this section. Summary results are shown in Tables 1, 1a and 1b as well as in Figures 1, 1a and 1b.

Figure 1 shows overall developments of the debt situation during 1980-2002. One can see clearly from there that there has been uneven growth in PV and DOD. To grasp these differences better, Figure 2 is intended to show growth rates of PV and DOD. Here we can clearly see the interesting picture where PV may grow but DOD falls and the other way around.

The differences in growth rates between DOD and PV are shown in Figure 1b. Here we can see drastic differences: from +15% to -8%. Such differences would amount to equally drastic consequences for modeling debt sustainability.

Numbers for individual regions are presented in tabular form in Tables 1, 1a and 1b. Table 1b showing the differences in growth rates between PV and DOD indicates a rather uniform picture across regions. The differences are the function of discount rates, currency composition, maturity and grace period. So, the uniformity is explained by the high correlation of developments on international financial markets.

As many economic models operate with differences, using PV instead of DOD may drastically change our understanding of the debt outlook, especially for severely indebted countries.

Even long-term nominal growth for PV and DOD shows substantial differences: during 1980-2002 DOD grew at 6.8% annually, whereas PV increased at 8.1% annually. Corrected for inflation, these numbers become 3.8% and 5.1%, respectively, this is equivalent to providing a different answer to the basic question: whether the debt burden relative to the GDP has increased or decreased during 1980-2001.

These estimates provide a significantly different picture of the debt burden dynamics in the developing countries. Using DOD instead of PV in debt analyses not only underestimates the severity of the debt problem, but often assumes
wrong dynamics for many crucial periods, which, in turn, will undermine the analyses’ outcomes in those countries.
Limit properties of the GE Function

Some of the limit properties of the GE function are interesting per se. Presented below are cases, where $i \to 0$ (interest rate approaching 0), $d \to 0$ (discount rate approaching 0), $n \to M$ (length of payments approaching maturity, i.e. grace approaching zero), $m \to 0$ (length of payments approaching 0, i.e. bullet payment).

\[
\text{Limit}[t, \, i \to 0]
\]
\[
\left( \frac{1}{1+d} \right)^{N} \left( \frac{1}{1+d} \right)^{-n-N} \cdot \frac{d m}{d^{2} m}
\]

\[
\text{Limit}[t, \, d \to 0]
\]
\[
\frac{1}{2} i \left( \frac{1}{1+d} \right)^{N} (1 + 2 N)
\]

\[
\text{Limit}[t, \, m \to N]
\]
\[
(d \cdot i) \left( \left( \frac{1}{1+d} \right)^{N} + \frac{d}{d^{2}} N \right)
\]

\[
\text{Limit}[t, \, m \to 0]
\]
\[
\left( \frac{1}{1+d} \right)^{N} \left[ \left( \frac{1}{1+d} \right)^{N} - e \right]
\]

Derivatives can be presented as follows:

\[
D[f[N, m, d, i], N]
\]
\[
\frac{\left( \frac{1}{1+d} \right)^{-m-N} \left( -1 + \left( \frac{1}{1+d} \right)^{m} \right) \cdot (d - i) \cdot \log \left( \frac{1}{1+d} \right)}{d^{2} m}
\]

\[
D[f[N, m, d, i], m]
\]
\[
\left( \frac{1}{1+d} \right)^{m-N} \left( -1 + \left( \frac{1}{1+d} \right)^{m} \cdot \log \left( \frac{1}{1+d} \right) \right)
\]

\[
D[f[N, m, d, i], i]
\]
\[
\left( \frac{1}{1+d} \right)^{N} \left( \frac{1}{1+d} \right)^{-m-N} \cdot d m
\]

\[
D[f[N, m, d, i], d]
\]
\[
\frac{1}{d^{3} m} \left( d \left( \frac{1}{1+d} \right)^{N} \left( \frac{1}{1+d} \right)^{-m-N} \cdot d m \right) + 2 (d - i) \left( \frac{1}{1+d} \right)^{N} \left( \frac{1}{1+d} \right)
\]

\[
\cdot \left( d - i \right) \left( m - \left( \frac{1}{1+d} \right)^{N} + \left( \frac{1}{1+d} \right)^{N} (-m + N) \right)
\]
Figure 1. Developing World DOD & PV, 1980-2002

Figure 1a. Growth rates of World DOD & PV, 1970-2002
Figure 1b. Difference between World DOD & PV growth rates, 1970-2002
**Table 1. Public and Publicly Guaranteed Debt, 1980-2001**

<table>
<thead>
<tr>
<th>REGION</th>
<th>Year</th>
<th>US$ millions</th>
<th>Per Cent</th>
</tr>
</thead>
<tbody>
<tr>
<td>Africa DOD</td>
<td>54,112</td>
<td>31,649</td>
<td>49,315</td>
</tr>
<tr>
<td>Asia NPV</td>
<td>54,112</td>
<td>31,649</td>
<td>49,315</td>
</tr>
<tr>
<td>Africa NPV</td>
<td>54,112</td>
<td>31,649</td>
<td>49,315</td>
</tr>
<tr>
<td>East&amp;Central Europe DOD</td>
<td>54,112</td>
<td>31,649</td>
<td>49,315</td>
</tr>
</tbody>
</table>

**Table 1a. Growth of Public and Publicly Guaranteed Debt, 1980-2001**

<table>
<thead>
<tr>
<th>REGION</th>
<th>Per Year</th>
</tr>
</thead>
<tbody>
<tr>
<td>Africa DOD</td>
<td>5.6%</td>
</tr>
<tr>
<td>Asia DOD</td>
<td>6.5%</td>
</tr>
<tr>
<td>Asia NPV</td>
<td>5.8%</td>
</tr>
<tr>
<td>East&amp;Central Europe DOD</td>
<td>6.5%</td>
</tr>
</tbody>
</table>

**Table 1b. Difference of DOD and PV Growth of Public and Publicly Guaranteed Debt, 1980-2001**

<table>
<thead>
<tr>
<th>REGION</th>
<th>Per Year</th>
</tr>
</thead>
<tbody>
<tr>
<td>Africa DOD</td>
<td>5.8%</td>
</tr>
<tr>
<td>Asia DOD</td>
<td>-1.7%</td>
</tr>
<tr>
<td>Asia NPV</td>
<td>-0.2%</td>
</tr>
<tr>
<td>East&amp;Central Europe DOD</td>
<td>5.3%</td>
</tr>
</tbody>
</table>

**Table 1c.**